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Comment on "Swimming at low Reynolds number of a cylindrical body in a circular tube" [Phys. Fluids 22, 113604 (2010)]

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The following comment presents a correction to Felderhof's work on confined microswimming.[1] In this work, Felderhof obtained an analytical solution for a long flagellum tail swimming in a confined cylinder and showed that the confinement entailed a slight mathematical complication relative to Taylor's original work of an unconfined swimmer.[2] In both these works, Taylor and Felderhof employed a perturbation scheme assuming a small surface (radial) displacement A, relative to the swimmer radius a, $(A/a \ll 1)$. They calculated the energy input $\overline{D}$ per unit length and per unit time that is required to sustain the planar and helical motion of a flagellum at a given propulsive velocity V.

Felderhof's work provides an important result, proving that the velocity and the required energy input of a swimmer in a tube is higher than that of a swimmer in an unbounded region for a given swimmer radius, oscillation wavenumber, k, and oscillation frequency, ω. For the case of a flagellum undergoing helical motion, Felderhof provides an incorrect expression for the rotational velocity of a confined swimmer.

Taylor[2] showed that, for a freely suspended swimmer, the fundamental solution for the homogenous azimuthal component of the Navier Stokes equation requires the addition of the term

$$V_t = \frac{\Omega}{r} + \Gamma r = \frac{\Omega k}{z} + \Gamma \frac{z}{k}, \tag{1}$$

where r stands for the radial coordinate and $z = rk$ is the dimensionless radial coordinate. The first term represents the fundamental irrotational solution while the latter is associated with rigid body rotation in which Γ is the angular velocity.

For the unconfined case, Taylor correctly (and trivially) assumed that $\Gamma = 0$, so that the tangential velocity decays to zero at infinity. Felderhof also assumed incorrectly that $\Gamma = 0$. This assumption does not satisfy the no-slip boundary condition at the outer bounding wall, namely, $V_t(z = \alpha a) = 0$. Here, α is the ratio between the confining tube and the flagellum radii. If the full expression (1) is accounted for, it can easily be shown that

$$\Omega = -\frac{A\alpha^2 z_1}{2(\alpha^2 - 1)} \left\{ \frac{dv_1}{dz} \right\}_{z=z_1}, \quad \Gamma = \frac{Ak^2}{2z_1(\alpha^2 - 1)} \left\{ \frac{dv_1}{dz} \right\}_{z=z_1}, \tag{2}$$

where $z_1 = ka$ is the dimensionless radius of the inner cylinder and v_1 is the first order solution for the velocity in the azimuthal direction. It is readily seen that $\Gamma \to 0$ for the unconfined case.

$(\alpha \to \infty)$ and Taylor's result is recovered while for finite values of α, Γ does not vanish. Consequently, the resultant torque

$$G = 4\pi\mu\Omega = -2\pi\mu A z_1 \alpha_F \left\{ \frac{dv_1}{dz} \right\}_{z=z_1} \tag{3}$$

needs to be modified by the factor

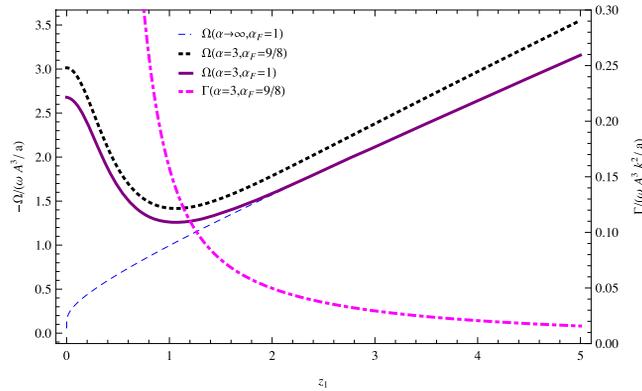$$\alpha_F = \frac{\alpha^2}{\alpha^2 - 1}, \tag{4}$$

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FIG. 1. $-\Omega/(\omega A^3/a)$ vs. the dimensionless radius z_1 for the unconfined and confined cases $\alpha \to \infty$ and $\alpha = 3$, respectively, and $\Gamma/(\omega A^3 k^2/a)$ for the confined case $\alpha = 3$.
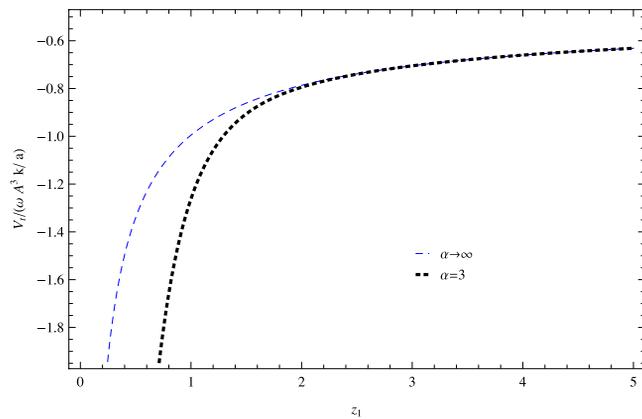

FIG. 2. $V_t/(\omega A^3 k/a)$ vs. the dimensionless radius z_1 for the unconfined case and confined cases with the correct α_F.

where μ stands for the viscosity.

In Fig. 1, we plot the dimensionless parameter $-\Omega/(\omega A^3/a)$ as a function of the dimensionless radius z_1 for three different cases; the unconfined geometry $\alpha \to \infty$ with $\alpha_F = 1$, the confined geometry of $\alpha = 3$ with the appropriate $\alpha_F = 9/8$ and the confined geometry $\alpha = 3$ with Felderhof's incorrect result $\alpha_F = 1$ which does not fully account for the confinement. (It corresponds to Fig. 8 in Ref. 1.) We also plot in Fig. 1 the dimensionless parameter $\Gamma/(\omega A^3 k^2/a)$ for the confined case $\alpha = 3$ for which Felderhof assumed incorrectly a zero value. It is clear that the correct solution predicts an additional increase of the required irrotational term as well as the torque input. Finally, we note here that the solution presented by Felderhof is twice as small as that presented here; however, this is likely due to a different normalization.

It can be seen that the irrotational term does not converge to Taylor's solution for large values of z_1. This is because the free body rotation term does not trivially vanish. In Fig. 2, we plot the normalized azimuthal velocity given by Eq. (1) and show that it is the sum of these two terms that converges to the solution of an unconfined case for large z_1 values.

In addition, we wish to point out two minor errors in Felderhof's work: The normalization of the required rate of energy input per unit length should read $\overline{D}/\mu\omega^2 A^2$ and the dimensionless efficiency should be $E = \mu\omega a V/\overline{D}$.

[1] B. U. Felderhof, "Swimming at low Reynolds number of a cylindrical body in a circular tube," Phys. Fluids **22**(11), 113604 (2010).
[2] G. I. Taylor, "The action of waving cylindrical tails in propelling microscopic organisms," Proc. R. Soc. A **211**(1105), 225–239 (1952).